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ADAPTIVE CONTROL BY OBSERVATIONS†

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The problem of a guaranteed estimate [1, 2] of an unknown finite-dimensional vector is considered. A method of formalizing the problem by the observation in Hilbert space is proposed. Using this formulation, it is shown that the solutions of the problems of programmed and adaptive control by the process of observation are identical. The paper touches on the investigations described in [3–6].

1. FORMULATION OF THE PROBLEM

WE WILL formulate [4] the problem of control by an observation process in Hilbert space. Suppose an unknown vector $x \in R^n$ is measured in accordance with the model

$$y = Ax + \xi \tag{1.1}$$

where the measured signal y and the unknown perturbation ξ are elements of a Hilbert space H with a scalar derivative $\langle \cdot, \cdot \rangle$ and $A \in L(R^n, H)$ is a given linear continuous operator, which will henceforth be called the measuring instrument.

The information on the perturbations ξ is completed by the inclusion

$$\begin{gathered} \xi \in \Xi = E(\alpha^{-2}I, 0), \quad \alpha > 0 \\ E(P, u_0) = \{u: \langle (u-u_0), \ P(u-u_0) \rangle \leqslant 1\} \end{gathered} \tag{1.2}$$

Here $E(P, u_0)$ is an ellipsoid in Hilbert (finite-dimensional) space with centre u_0 and self-conjugate non-negative operator (matrix) P, and I is the identity operator.

The minimax of guaranteed approach to the estimate of the vector x is based on a determination of the information set $X(y)$ of the parameters, combined with the measurement of y. In the case considered $X(y)$ is an ellipsoid in R_n [2, 7, 8]

$$\begin{gathered} X(y) = E(P(y), x_0), \quad P(y) = (\alpha^2 - h^2(y))^{-2} P \\ P = A^*A, \quad Px_0 = A^*y, \quad h^2(y) = \langle y, y \rangle - (x_0, A^*y) = \langle y, y \rangle - \langle Ax_0, y \rangle \end{gathered} \tag{1.3}$$

Here $(\cdot, \cdot)$ is the scalar product in R^n while A^* is the conjugate operator, so that $\langle y, Ax \rangle = (A^*y, x)$ for any $y \in H$, $x \in R^n$.

The quantity $h(y)$ is identical with the norm of the projection of the element y (or, which is equivalent, ξ) onto the subspace $\ker A^* = \{h \in H: A^*h = 0\}$. It is obvious that $0 \leqslant h(y) \leqslant \alpha$, and when $h(y) = \alpha$ the information set $X(y)$ degenerates to a point.

Assuming that we choose the centre of the ellipsoid (1.3) as the estimate of the vector x, the accuracy of the estimation is naturally characterized by a function of $P(y)$. Suppose $\Phi(\cdot)$ is a specified function of the matrix argument and $F(A, y) = \Phi(P(y))$. We will assume, in addition, that the set $\Sigma \subset L(R^n, H)$ of pos-

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sible measuring instruments is defined, and it is necessary to determine the element $A_0 \in \Sigma$, which optimizes the accuracy of the estimation. To formalize this problem it is necessary, however, to set up a criterion which is independent of the measurement of y, which cannot be known before choosing A_0. By the logic of the guaranteed approach, the required convolution can be obtained by maximizing $F(A, y)$ with respect to y in the set of possible outcomes (1.1) and (1.2). Hence, we obtain the following problem of choosing the optimum programmed measuring instrument

$$\max_y F(A, y) = \max_y \Phi(P(y)) \Rightarrow \min, \qquad A \in \Sigma \tag{1.4}$$

In (1.4) we have optimized the accuracy of the estimation in calculating the worst realization of the perturbations, which, in practice, is equivalent to an *a priori* approach to the estimation [1, 2] and justifies the use of the term "programmed".

We will now assume that the measurement of (1.1) and the choice of A_0 can be represented in the form of processes which have a certain duration. It then makes sense to speak of the problem of adaptive optimization of measurements depending on the incoming information.

Before we formulate the problem in an arbitrary Hilbert space, we will illustrate it using the example of the space $H = L_2$, [0, 1].

Suppose

$$y(t) = (a(t), x) + \xi(t), \qquad t \in [0, 1]$$

$$\int_0^1 \xi^2(t)\, dt \leqslant \alpha^2, \qquad a(\cdot) \in L_2^n[0, 1]$$

Here

$$P = \int_0^1 a(t) a^T(t)\, dt$$

If we know that in the interval $[0, \tau]$, $0 \leqslant \tau \leqslant 1$

$$\begin{aligned} &a(t) = a_0(t), \qquad y(t) = y_0(t) \\ &a_0(\cdot) \in L_2^n[0, 1], \qquad y_0(\cdot) \in L_2[0, 1] \end{aligned} \tag{1.5}$$

we can say that we have chosen the permissible measuring instrument which minimizes the function

$$F_\tau(A) = \max_{y(\cdot)} F(A, y(\cdot))$$

where the minimum and maximum are calculated for condition (1.5). Solving this problem for each τ we can set up adaptive control of the measurements.

Before considering an arbitrary Hilbert space we note that relations (1.5) can be regarded as a specification of the projections of the elements $a(\cdot)$ and $y(\cdot)$ on a certain subspace.

Suppose we have determined the expansion of unity [9, p. 352] into H, i.e. the family of projectors $\{T_\tau, \tau \in [0, 1]\} \subset L(H, H)$, $T_0 = 0$, $T_1 = I$, $T_\tau \leqslant T_\vartheta$, $\tau \leqslant \vartheta$. The parameter $\tau \in [0, 1]$ will be identified with time and we will assume that up to the instant τ the signal $y_\tau = T_\tau y$ corresponding to the chosen measuring instrument $A_\tau = T_\tau A$ has been realized. We will also assume that no other information arrives regarding the perturbations. Then the family of problems of adaptive control of measurements can be written as follows:

$$\max_y \{F(A, y) : T_\tau y = y_\tau\} \Rightarrow \min; \quad A \in \Sigma, \quad T_\tau A = A_\tau^\circ \tag{1.6}$$

It is obvious that problems (1.4) and (1.6) are identical when $\tau = 0$.

Proceeding in the same way as in the theory of the planning of an experiment, problem (1.6) can be regarded as a realization of the idea of successive planning [10]. It is precisely this idea which was put forward, for example, in [5], as the basis for taking into account *a posteriori* information. In this connection, it is quite unexpected that in the case considered the solution of problem (1.6) contains no new information on the optimum measuring instruments.

2. THE MAIN RESULT

We will assume that the function $\Phi(\cdot)$, representing the error of the estimation, satisfies the following condition

$$\Phi(\lambda P) = r(\lambda)\,\Phi(P) \tag{2.1}$$

for any $\lambda > 0$ and a certain monotonically non-decreasing function $r(\cdot)$. All the generally accepted criteria of the accuracy of estimation satisfy this condition. For example, for $\Phi(P) = \det(P^{-1})$, representing the volume of the ellipsoid $E(P, 0)$, $r(\lambda) = \lambda^{-n}$.

Theorem. Suppose condition (2.1) holds, A_0 is a solution of problem (1.4), and in (1.6) $A_\tau^0 = T_\tau A_0$. Then the solutions of problems (1.4) and (1.6) of programmed and adaptive control of the process of observation are identical.

Proof. We will first describe the general scheme of the proof. In the space $\Omega = R^n \times H$ of parameters $w = (x, \xi)$ relations (1.1) and (1.2) can be written in the form

$$\begin{aligned} &y = Dw, \qquad w \in W_0, \\ &D \in L(\Omega, H), \qquad Dw = Ax + \xi, \qquad W_0 = R^n \times \Xi = E(Q, 0) \end{aligned} \tag{2.2}$$

Suppose $W_\tau = \{w \in W_0 : y_\tau = T_\tau Dw\}$ is the information set of parameters $w \in \Omega$, compatible with the known signal y_τ.

Basing ourselves on the fact that no additional information on the perturbations arrives, we can assume that $T_\tau \xi \in \Xi$.

Suppose $W_\tau(y)$ is the information set in (2.2) when W_0 is replaced by W_τ, while $X_\tau(y)$ is the projection of $W_\tau(y)$ onto R^n. It is obvious that in the case considered W_τ, $W_\tau(y)$ and $X_\tau(y)$ are ellipsoids. If $X_\tau(y) = E(P_\tau(y),\ x_\tau^0)$, problem (1.6) can be represented in the form

$$\max_{y \in DW_T} \Phi(P_T(y)) \Rightarrow \min;\ A \in \Sigma_T = \{A \in \Sigma,\ T_T A = A_T^0\} \tag{2.3}$$

It turns out that

$$\max\{\Phi(P_T(y)) : y \in DW_T\} = \rho^{-2}(y_T)\,\Phi(P) \tag{2.4}$$

where $\rho(\cdot)$ is a certain scalar function.

Hence, the functionals in (2.3) and (1.4) differ solely in a scalar factor which depends on the prehistory (of the signal up to the instant τ). This fact also establishes the correctness of the theorem.

We will now justify (2.4). W_τ is the intersection of the ellipsoid W_0 with the affine plane of the solutions of the equation

$$T_T Dw = D_T w = y_T \tag{2.5}$$

We will put

$$L_T = \ker D_T = \{w \in \Omega : D_T w = 0\}$$

Lemma. In (x, ξ) coordinates, the form of the operator Γ_τ of the orthogonal projection onto the subspace L_τ is given by the expression

$$\Gamma_T = \left\| \begin{matrix} M_T & -M_T A_T^* \\ -M_T A_T^* & A_T M_T A_T^* + (I - T_T) \end{matrix} \right\|, \qquad M_T = (I + A_T^* A_T)^{-1} \tag{2.6}$$

Proof. By the definition of the operator of orthogonal projection for $w = (x, \xi) \in \Omega$

$$\langle w - \Gamma_T w, w - \Gamma_T w \rangle = \min_{u \in L_T} \langle w - u, w - u \rangle \tag{2.7}$$

If $u = (q, \epsilon) \in \Omega$, $u \in L_\tau$, then $A_\tau q + T_\tau \epsilon = 0$.

For fixed q, the general solution for ϵ of this equation can be represented in the form

$$\epsilon_T = -A_T q + (I - T_T)\sigma, \quad \sigma \in H \tag{2.8}$$

Consequently, finding Γ_τ is equivalent to minimizing the quadratic form

$$B(q, \sigma) = (x - q, x - q) + \langle \xi - \epsilon_T, \xi - \epsilon_T \rangle = (x, x) + \langle \xi, \xi \rangle +$$
$$+ (q, (I + A_T^* A_T) q) + \langle \sigma, (I - T_T)\sigma \rangle - 2(q, x - A_T^* \xi) - 2\langle \xi, (I - T_T)\sigma \rangle$$

Writing the necessary conditions for a minimum of $B(q, \sigma)$ with respect to q and σ, we obtain

$$(I + A_T^* A_T) q - (x - A_T^* \xi) = 0, \qquad (I - T_T)\sigma - (I - T_T)\xi = 0$$

Hence it follows that a minimum of $B(\cdot, \cdot)$ is reached at the point

$$q_T = (I + A_T^* A_T)^{-1}(x - A_T^* \xi), \qquad \sigma_T = (I - T_T)\xi \tag{2.9}$$

From (2.7) and (2.8) we obtain

$$\Gamma w = (q_T, \epsilon_T), \quad \epsilon_T = -A_T (I + A_T^* A_T)^{-1}(x - A_T^* \xi) + (I - T_T)\xi \tag{2.10}$$

Relations (2.9) and (2.10) are equivalent to (2.6). This proves the lemma.

Suppose now that x_τ is an arbitrary solution of the system of equations

$$A_T^* A_T x_T = A_T^* y_T$$

(If $A_\tau^* A_\tau$ is a non-degenerate matrix, then $x_\tau = (A_\tau^* A_\tau)^{-1} A_\tau^* y_\tau$.) Then $w_\tau = (x_\tau, y_\tau - A_\tau x_\tau) \in \Omega$ is a solution of Eq. (2.5) and $\Gamma_\tau Q w_\tau = 0$.

In fact, using the coordinate representations of the operators Q (see (1.2) and (2.2)) and Γ_τ (see (2.6)), we obtain

$$\Gamma_T Q w_T = (M_T (A_T^* A_T x_T - A_T^* y_T), \quad A_T M_T (A_T^* A_T x_T - A_T^* y_T)) = 0$$

Representing the general solution of Eq. (2.5) in the form $w = w_\tau + v$, $v \in L_\tau$, we obtain

$$W_T = \{ w = w_T + v, \quad v \in L_T, \quad v \in V_T \} \tag{2.11}$$
$$V_T = E(\beta^{-2}(y_T) Q, 0) = \beta(y_T) W_0$$
$$\beta^2(y_T) = (1 - \langle w_T, Q w_T \rangle) = (1 - \langle y_T - A_T x_T, y_T - A_T x_T \rangle)$$

We will introduce the variables

$$f = y - y_T + (I - T_T) A x_T, \quad z = x - x_T, \quad \eta = \xi - y_T + A_T x_T$$

For $v=(z, \eta)\in\Omega$ we have

$$T_T Dv = A_T (x-x_T) + T_T \xi - y_T + A_T x_T = A_T x + T_T \xi - y_T = 0$$

Consequently, $v=(z, \eta)\in L_\tau$, and by virtue of (2.2), (2.6) and (2.11), $X_\tau(y)$ is identical with the information set $Z(f)$ for the system

$$f = Az + \eta, \quad \eta\in E(\gamma^{-2}I, 0), \quad \gamma=\gamma(y_T)=\alpha\beta(y_T)$$

Relation (2.4) hence follows from (1.3) and (2.1). This proves the theorem.

The content of the theorem can be illustrated using the following very simple example. Suppose two measurements were made of an unknown number x

$$y=(y_1, y_2)\in R^2, \quad y_i = a_i x + \xi_i, \quad i=1,2, \quad \xi_1^2+\xi_2^2\leqslant 1$$

It is clear that the length of the section $X(y)$ depends very much on the values of y_1 and y_2 obtained. However, the dimensions of the ellipsoid obtained by the section of the cylinder

$$W_0 = \{(x, \xi_1, \xi_2)\in R^3: \xi_1^2+\xi_2^2\leqslant 1\}$$

with the plane

$$\Pi_1 = \{(x, \xi_1, \xi_2)\in R^3: y_1 = a_1 x + \xi_1\}$$

is obviously independent of y_1.

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